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$H(\mathrm{div})$ conforming finite element methods for the coupled Stokes and Darcy problem☆

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ABSTRACT

This paper proposes and analyzes a numerical method for solving the coupled Stokes and Darcy problem, an interface problem between a fluid, governed by Stokes equations, and a flow in a porous medium, governed by Darcy equations. The method employs $H(\mathrm{div})$ conforming finite elements for the velocity field in both Stokes and Darcy subdomains. Optimal-order error estimates are established.

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1. Introduction

We consider the coupled system of Stokes and Darcy equations. This model can be used for the simulation of the transport of contaminants through rivers into the aquifers. Let Ω_1 and Ω_2 be two disjoint open bounded subsets of $\Omega \subset \mathbb{R}^2$, with Lipschitzian boundaries $\partial\Omega_1$ and $\partial\Omega_2$. We denote by $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$ the interface between the subdomains Ω_1 and Ω_2. Let $\mathbf{n}_{12}$ be the unit normal vector to Γ_{12} directed from Ω_1 to Ω_2 and let $\boldsymbol{\tau}_{12}$ be the unit tangent vector on Γ_{12}. The remaining parts of the boundaries are $\Gamma_i = \partial\Omega_i \setminus \Gamma_{12}$ for $i = 1, 2$. We assume that Γ_{12} is a polygonal line (see Fig. 1).

Denote by $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$ the fluid velocity and by $p = (p_1, p_2)$ the fluid pressure, where $\mathbf{u}_i = \mathbf{u}|_{\Omega_i}$ and $p_i = p|_{\Omega_i}$. In the fluid domain Ω_1, we assume that the flow is governed by the stationary Stokes equations

$$-\nabla \cdot (2\mu\mathbf{D}(\mathbf{u}_1) - p_1\mathbf{I}) = \mathbf{f}_1 \quad \text{in } \Omega_1, \tag{1.1}$$

$$\nabla \cdot \mathbf{u}_1 = 0 \quad \text{in } \Omega_1, \tag{1.2}$$

$$\mathbf{u}_1 = \mathbf{0} \quad \text{on } \Gamma_1, \tag{1.3}$$

where $\mathbf{D}(\mathbf{u}_1) := \frac{1}{2}(\nabla\mathbf{u}_1 + (\nabla\mathbf{u}_1)^T)$ is the deformation rate tensor and $\mu > 0$ is the kinematic viscosity of the fluid. Space averaged velocity and pressure in the porous domain Ω_2 are governed by Darcy equations

$$\mathbf{u}_2 = -\mathbf{K}\nabla p_2 \quad \text{in } \Omega_2, \tag{1.4}$$

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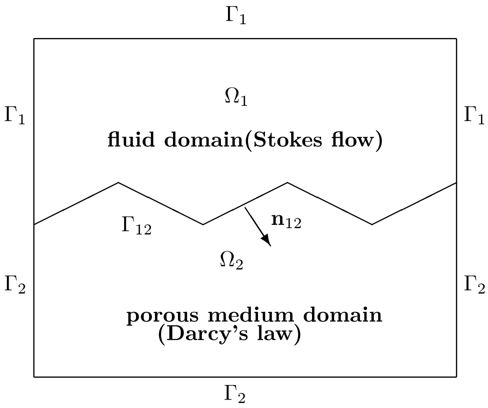

Fig. 1. Example of domain.

$$\nabla \cdot \mathbf{u}_2 = f_2 \quad \text{in } \Omega_2, \tag{1.5}$$
$$\mathbf{u}_2 \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_2, \tag{1.6}$$

where the symmetric, positive definite tensor $\mathbf{K}$ is bounded below and above uniformly, for some $0 < k_0 \le k_1 < \infty$,

$$k_0 \xi^T \xi \le \xi^T \mathbf{K}(x) \xi \le k_1 \xi^T \xi, \qquad \forall x \in \Omega_2, \quad \forall \xi \in \mathbb{R}^n. \tag{1.7}$$

The source f_2 is assumed to satisfy the solvability condition

$$\int_{\Omega_2} f_2 \mathrm{d}x = 0. \tag{1.8}$$

Note that if Eq. (1.4) holds then $\mathbf{u}_2$ can be eliminated and Eq. (1.5) takes the form of a second-order elliptic equation

$$\nabla \cdot (-\mathbf{K} \nabla p_2) = f_2 \quad \text{in } \Omega_2. \tag{1.9}$$

On the interface we consider the transmissibility conditions

$$\mathbf{u}_1 \cdot \mathbf{n}_{12} = \mathbf{u}_2 \cdot \mathbf{n}_{12}, \tag{1.10}$$
$$p_1 - 2\mu (\mathbf{D}(\mathbf{u}_1)\mathbf{n}_{12}) \cdot \mathbf{n}_{12} = p_2, \tag{1.11}$$
$$\mathbf{u}_1 \cdot \boldsymbol{\tau}_{12} = -2G(\mathbf{D}(\mathbf{u}_1)\mathbf{n}_{12}) \cdot \boldsymbol{\tau}_{12}. \tag{1.12}$$

Here $G > 0$ is a friction constant that can be determined experimentally. Note that conditions (1.10) and (1.11) express mass conservation and equilibrium of normal forces across the interface Γ_{12}, respectively. The Beaver–Joseph–Saffman law (1.12) is the most accepted condition [1–3].

This coupled Stokes–Darcy problem, (1.1)–(1.8) and (1.10)–(1.12), has been studied from mathematical theory and numerical analysis viewpoints [4–9,3,10–12]. Finite element approximations firstly differ in the weak formulation of the coupled problem. While Stokes equations are generally handled using the standard mixed formulation, several approaches have been used for Darcy equations. The method presented in [3] employs mixed finite element discretizations for both parts of Stokes domain and Darcy domain. Since the stable families of finite elements for Stokes problem and Darcy problem usually are not the same, this approach easily leads to finite element discretizations with different choice of spaces for the two domains. Standard Stokes elements such as Taylor–Hood elements and MINI element are used for Stokes domain, and RT, BDM or BDDF elements are used for Darcy domain. A similar approach was presented in [8], but with the difference that the Bernardi–Raugel element is used for Stokes domain. [12] used a standard formulation for Stokes equations and a Galerkin least-squares formulation for a mixed form of Darcy equations. The method discussed in [6] is based on a standard finite element method for the equivalent second-order elliptic Eq. (1.9). In [9], flows are governed by the linear stationary Stokes system on one part of the domain and by a second-order elliptic equation derived from the Darcy law in the rest of the domain. Different from the methods mentioned above, the method in [9] solves the coupled problem by using standard Stokes elements like MINI element or Taylor–Hood element in the entire domain. In addition, Crouzeix–Raviart element for the velocities and piecewise constants for the pressures in both domains, combined with two stabilization terms penalizing the jumps of the discontinuous velocities over the edges, are employed in [4]. In particular, a stabilized continuous piecewise linear/piecewise constant method with an added penalization of pressure jumps over the edges was proposed in [5] for the Stokes–Darcy problem. Lower-order rectangular finite element methods for the singularly perturbed Stokes problem was proposed in [13]. In [10,11], DG methods were used for the coupled Stokes and Darcy problem. [14] analyzed a priori and a posteriori estimates for some classes of methods for the parameter-dependent Brinkman problem, which covers a field of problems from Darcy equations to Stokes equations. In [15], a finite difference streamline diffusion nonconforming finite element approximation was proposed for solving the time-dependent linearized Navier–Stokes equations.

The commonly used conforming finite element methods for the Stokes domain are based on a variational formulation which is obtained by testing the momentum Eq. (1.1) by functions in $H^1_{\Gamma_1}(\Omega_1)^2$ and the continuity Eq. (1.2) in $L^2(\Omega_1)$ (see Section 2 for their definitions). The corresponding finite element method requires a pair of finite element spaces which are conforming in $H^1 \times L^2$ and satisfy the inf – sup condition. These constraints usually result in finite element approximations, denoted by $(\mathbf{u}_{1h}, p_{1h})$, which do not satisfy the following divergence-free equation cellwise

$$\nabla \cdot \mathbf{u}_{1h}(\mathbf{x}) = 0 \quad \forall \mathbf{x} \in \Omega_1. \tag{1.13}$$

This equation requires the numerical solution $\mathbf{u}_{1h}$ to be a member of the Sobolev space $H(\text{div}; \Omega_1)$. In this sense, DG methods [10,11] may not be appropriate when (1.13) is needed. On the other hand, the $H^1 \times L^2$ conforming finite element methods require the total continuity $\mathbf{u}_{1h}$, which is beyond what is required for satisfaction of (1.13). Therefore, it appears that the $H(\text{div})$ elements might be appropriate for approximating the solution of the Stokes domain [16,17].

In [16], a finite element scheme for the Stokes equations was derived and analyzed by using existing $H(\text{div})$ finite elements of the Raviart–Thomas type. [17] extended the results of [16] to the Navier–Stokes equations. In [18], three different methods were presented to construct uniformly stable finite element approximation for 2D and 3D Darcy–Stokes interface problems. Those equations are related to Brinkman model that treated both the Darcy law and Stokes equations in a single form of partial differential equations but with strongly discontinuous viscosity coefficient and zeroth-order term coefficient. Among the three methods, one is to construct uniformly stable elements by modifying some well-known $H(\text{div})$ conforming elements. [19] discussed a robust and nonconforming finite element method for a family of singular perturbation problems in two space dimensions. A key issue related to such nonconforming approximations of H^1 vector fields is whether Korn's inequality holds for the discrete spaces. In [20], Korn's inequalities for piecewise H^1 vector fields were established, which could be applied to classical nonconforming finite element methods, mortar methods and discontinuous Galerkin methods. [21] strengthened a general result of Brenner [20] on Korn's inequality for nonconforming finite element methods and showed that a robust Darcy–Stokes element satisfies Korn's inequality.

The goal of this paper is to continue the investigation in $H(\text{div})$ finite element methods by extending the results of [16,17] to the coupled Stokes and Darcy problem.

The rest of this paper is organized as follows. Section 2 introduces some preliminaries and notations for Sobolev spaces. Section 3 presents $H(\text{div})$ conforming finite element methods for the coupled Darcy and Stokes problem. Finite element approximations of the coupled problem is presented in Section 4. The a prior error estimates are established in Section 5. Finally conclusions follow.

Throughout the paper, vector-valued functions are written in boldface. We employ $\mathbf{0}$ to denote a generic null vector and use C and c, with or without subscripts, to denote generic constants independent of the mesh size h, which may take different values at different places.

2. Preliminaries and notations

For $i = 1, 2$, let $\mathcal{T}_h^i$ be a shape regular simplicial triangulation of the domain Ω_i, consisting of triangles of maximum diameter h_i. Define $\mathcal{T}_h$ by $\mathcal{T}_h|_{\Omega_i} = \mathcal{T}_h^i$. Let Γ_h^i be the set of interior edges of Ω_i, and define Γ_h by $\Gamma_h|_{\Omega_i} = \Gamma_h^i$. Along the interface Γ_{12}, the two meshes $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ are related in the following sense: any edge $e = \partial K_1 \cap \Gamma_{12}$ belongs to only one element $K_2 \in \mathcal{T}_h^2$ with $K_1 \in \mathcal{T}_h^1$.

For any nonnegative integer k, we use the classical definitions for the Sobolev spaces $H^k(D)$ on a domain $D \subset \mathbb{R}^2$.

$$H^k(D) = \{v \in L^2(D),\ \forall |m| \le k, \partial^m v \in L^2(D)\},$$

with the usual notations

$$m = (m_1, m_2), \qquad |m| = m_1 + m_2, \qquad \partial^m = \partial_{x_1}^{m_1} \partial_{x_2}^{m_2}.$$

The associated seminorm $|\cdot|_{k,D}$ and the norm $\|\cdot\|_{k,D}$ are given by

$$|v|_{k,D} = \left(\sum_{|\alpha|=k} \int_D |\partial^\alpha v|^2 \mathrm{d}D \right)^{\frac{1}{2}},$$

and

$$\|v\|_{k,D} = \left(\sum_{j=0}^{k} |v|_{k,D}^2 \right)^{\frac{1}{2}}.$$

If $D = \Omega$, we shall drop the subscript D in the norm and seminorm. The space $H_0^k(\Omega)$ denotes the closure in $H^k(\Omega)$ of the set of the infinitely differentiable functions with compact supports in Ω. For the corresponding n-dimensional vector spaces, we put superscript n on the scalar notation, such as, $H_0^k(\Omega)^n$ and $H^k(\Omega)^n$. The notation $L_0^2(\Omega)$ denotes the space of $L^2(\Omega)$

with mean value zero. We use $\langle\cdot,\cdot\rangle$ to denote the L^2 inner product as well as the duality pairing between H_0^k and its dual space. Let

$$H^1_{\Gamma_1}(\Omega_1)^2 = \{\mathbf{v}_1 \in H^1(\Omega_1)^2 : \mathbf{v}_1|_{\Gamma_1} = \mathbf{0}\},$$

be the space used in Section 1. The space $H(\text{div};\Omega)$ is defined as the set of vector-valued functions on Ω which, together with their divergence, are square integrable, i.e.,

$$H(\text{div};\Omega) = \{\mathbf{v} \in L^2(\Omega)^2;\ \nabla\cdot\mathbf{v} \in L^2(\Omega)\},$$

and equipped with the norm

$$\|\mathbf{v}\|_{H(\text{div};\Omega)} = (\|\mathbf{v}\|^2_{\Omega} + \|\nabla\cdot\mathbf{v}\|^2_{\Omega})^{1/2}.$$

It is well known that for all $\mathbf{v}_i \in H(\text{div};\Omega_i)$, $\mathbf{v}_i\cdot\mathbf{n}_i \in H^{-1/2}(\partial\Omega_i)$. The restriction of $\mathbf{v}_i\cdot\mathbf{n}_i$ to Γ_i, however, may not lie in $H^{-1/2}(\Gamma_i)$. We define the space X for the velocity as

$$X := \{\mathbf{v}_i \in H(\text{div};\Omega_i) : \mathbf{v}_i|_K \in H^1(K)^2,\ \forall K \in \mathcal{T}_h^i,\ \langle\mathbf{v}_i\cdot\mathbf{n}_i, w\rangle_{\partial\Omega_i} = 0,\ \text{for all } w \in H^1_{0,\Gamma_{12}}(\Omega_i)\},$$

with

$$H^1_{0,\Gamma_{12}}(\Omega_i) = \{w \in H^1(\Omega_i) : w = 0 \text{ on } \Gamma_{12}\}.$$

The space Q for the pressure is defined by

$$Q := \left\{q_i \in L^2(\Omega_i) : \int_{\Omega_1} q_1 + \int_{\Omega_2} q_2 = 0\right\}.$$

Finally, we introduce some notations associated with traces. Let ϕ be a piecewise smooth scalar or vector function. For each edge e of the triangles in $\mathcal{T}_h^1 \cup \mathcal{T}_h^2$, we fix a unit normal vector denoted $\mathbf{n}_e$. If the edge e is a boundary edge, the vector $\mathbf{n}_e$ coincides with the unit normal vector exterior to Ω. For any two triangles K_i and K_j (with $i \neq j$) that share a common edge e, the average function $\{\phi\}$ and jump function $[\phi]$ of ϕ are uniquely defined (see for example [10,11])

$$\{\phi\} := \frac{1}{2}(\phi|_{K_i} + \phi|_{K_j}), \qquad [\phi] := \phi|_{K_i} - \phi|_{K_j}.$$

On a boundary edge, we have $\{\phi\} := \phi$, $[\phi] := \phi$.

3. Weak formulations

Beginning with a classical solution of (1.1), multiplying (1.1) by testing functions $\mathbf{v}_1 \in X$, integrating by parts over one element $K \in \mathcal{T}_h^1$, and summing over all elements in $\mathcal{T}_h^1$, we have

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h^1}\int_K(-p_1\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}_1)) : \mathbf{D}(\mathbf{v}_1) - \sum_{e\in\Gamma_h^1}\int_e[(-p_1\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}_1))\mathbf{n}_e\cdot\mathbf{v}_1] \\
&\quad - \sum_{e\in\Gamma_{12}}\int_e(-p_1\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}_1))\mathbf{n}_{12}\cdot\mathbf{v}_1 - \sum_{e\in\Gamma_1}\int_e(-p_1\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}_1))\mathbf{n}_e\cdot\mathbf{v}_1 \\
&= \int_{\Omega_1}\mathbf{f}_1\mathbf{v}_1, \quad \forall\mathbf{v}_1 \in X.
\end{aligned} \tag{3.1}$$

Let $\boldsymbol{\tau}_e$ be the tangential direction to edge e so that $\mathbf{n}_e$ and $\boldsymbol{\tau}_e$ form a right-hand coordinate system. From the representation

$$\mathbf{v} = (\mathbf{v}\cdot\mathbf{n})\mathbf{n} + (\mathbf{v}\cdot\boldsymbol{\tau})\boldsymbol{\tau},$$

we decompose $\mathbf{v}$ and $\mathbf{D}(\mathbf{u}_1)\,\mathbf{n}_e$ into their normal and tangential components

$$\begin{aligned}
\mathbf{D}(\mathbf{u}_1)\mathbf{n}_e\cdot\mathbf{v} &= (((\mathbf{D}(\mathbf{u}_1)\mathbf{n}_e)\cdot\mathbf{n}_e)\mathbf{n}_e + ((\mathbf{D}(\mathbf{u}_1)\mathbf{n}_e)\cdot\boldsymbol{\tau}_e)\boldsymbol{\tau}_e)\cdot((\mathbf{v}\cdot\mathbf{n}_e)\mathbf{n}_e + (\mathbf{v}\cdot\boldsymbol{\tau}_e)\boldsymbol{\tau}_e) \\
&= ((\mathbf{D}(\mathbf{u}_1)\mathbf{n}_e)\cdot\mathbf{n}_e)(\mathbf{v}\cdot\mathbf{n}_e) + ((\mathbf{D}(\mathbf{u}_1)\,\mathbf{n}_e)\cdot\boldsymbol{\tau}_e)(\mathbf{v}_1\cdot\boldsymbol{\tau}_e).
\end{aligned}$$

Note that $p_1\mathbf{I}\mathbf{n}_e\cdot\mathbf{v} = p_1\mathbf{n}_e\cdot\mathbf{v}$. By the equality $[ab] = [a]\{b\} + \{a\}[b]$ and the regularity of the true solution, the interior integral is reduced to

$$-\sum_{e\in\Gamma_h^1}\int_e[(-p_1\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}_1))\,\mathbf{n}_e\cdot\mathbf{v}_1] = -2\mu\sum_{e\in\Gamma_h^1}\int_e\{(\mathbf{D}(\mathbf{u}_1)\mathbf{n}_e)\cdot\boldsymbol{\tau}_e\}[\mathbf{v}\cdot\boldsymbol{\tau}_e].$$

Here, we have used the fact that $\mathbf{v}_1 \in X$, which implies that $\mathbf{v}_1 \cdot \mathbf{n}_e$ is continuous across each interior boundary. With the interface conditions (1.11) and (1.12), the integral becomes

$$
\begin{aligned}
-\sum_{e\in\Gamma_{12}} \int_e (-p_1\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}_1))\mathbf{n}_{12}\cdot\mathbf{v}_1 &= \sum_{e\in\Gamma_{12}} \int_e (p_1 - 2\mu(\mathbf{D}(\mathbf{u}_1)\mathbf{n}_{12})\cdot\mathbf{n}_{12})(\mathbf{v}_1\cdot\mathbf{n}_{12}) - 2\mu((\mathbf{D}(\mathbf{u}_1)\mathbf{n}_{12})\cdot\boldsymbol{\tau}_{12})(\mathbf{v}_1\cdot\boldsymbol{\tau}_{12}) \\
&= \sum_{e\in\Gamma_{12}} \int_e p_2(\mathbf{v}_1\cdot\mathbf{n}_{12}) + \frac{\mu}{G}(\mathbf{u}_1\cdot\boldsymbol{\tau}_{12})(\mathbf{v}_1\cdot\boldsymbol{\tau}_{12}).
\end{aligned}
$$

Using the fact that $\mathbf{v}_1 \in X$ and the regularity of the true solution, we have

$$
-\sum_{e\in\Gamma_1} \int_e (-p_1\mathbf{I} + 2\mu\mathbf{D}(\mathbf{u}_1))\mathbf{n}_e\cdot\mathbf{v}_1 = -\sum_{e\in\Gamma_1} \int_e 2\mu((\mathbf{D}(\mathbf{u}_1)\mathbf{n}_e)\cdot\boldsymbol{\tau}_e)(\mathbf{v}_1\cdot\boldsymbol{\tau}_e).
$$

Noticing that $\mathbf{I} : \nabla\mathbf{v} = \nabla\cdot\mathbf{v}$ and substituting the above three equalities into Eq. (3.1), we have

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h^1} \int_K 2\mu\mathbf{D}(\mathbf{u}_1) : \mathbf{D}(\mathbf{v}_1) - \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e 2\mu\{\mathbf{D}(\mathbf{u}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\mathbf{v}_1\cdot\boldsymbol{\tau}_e] \\
&\quad + \sum_{e\in\Gamma_{12}} \int_e \frac{\mu}{G}(\mathbf{u}_1\cdot\boldsymbol{\tau}_{12})(\mathbf{v}_1\cdot\boldsymbol{\tau}_{12}) - \int_{\Omega_1} p_1\nabla\cdot\mathbf{v}_1 + \sum_{e\in\Gamma_{12}} \int_e p_2\mathbf{v}_1\cdot\mathbf{n}_{12} \\
&= \int_{\Omega_1} \mathbf{f}_1\cdot\mathbf{v}_1, \quad \forall\mathbf{v}_1 \in X.
\end{aligned}
$$

As in the usual DG methods, we further stabilize the above equation by adding the following term to its left-hand side

$$
S_\delta(\mathbf{u}_1,\mathbf{v}_1) := \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e 2\mu(\alpha h_e^{-1}[\mathbf{u}_1\cdot\boldsymbol{\tau}_e][\mathbf{v}_1\cdot\boldsymbol{\tau}_e] + \delta\{\mathbf{D}(\mathbf{v}_1)\,\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\mathbf{u}_1\cdot\boldsymbol{\tau}_e]),
$$

where h_e is the length of the edge e, $\alpha > 0$ is a stabilization parameter and $\delta \in \{1, -1\}$ is a symmetrization parameter. It is easy to see that $S_\delta(\mathbf{u}_1, \mathbf{v}_1) = 0$ for any $\mathbf{u}_1 \in H^1_{0,\Gamma_1}(\Omega_1)^2$ and $\mathbf{v}_1 \in X$. It follows from (1.2) that

$$
\int_{\Omega_1} q_1\nabla\cdot\mathbf{u}_1 = 0, \quad \forall q_1 \in Q.
$$

For simplification, we introduce two bilinear forms

$$
\begin{aligned}
a_{1,\delta}(\mathbf{u}_1,\mathbf{v}_1) := &\sum_{K\in\mathcal{T}_h^1} \int_K 2\mu\mathbf{D}(\mathbf{u}_1) : \mathbf{D}(\mathbf{v}_1) + S_\delta(\mathbf{u}_1,\mathbf{v}_1) \\
&- \sum_{e\in\Gamma_h^1\cup\Gamma_1} \int_e 2\mu\{\mathbf{D}(\mathbf{u}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\mathbf{v}_1\cdot\boldsymbol{\tau}_e] + \sum_{e\in\Gamma_{12}} \int_e \frac{\mu}{G}(\mathbf{u}_1\cdot\boldsymbol{\tau}_{12})(\mathbf{v}_1\cdot\boldsymbol{\tau}_{12}),
\end{aligned}
$$

and

$$
b_1(\mathbf{v}_1, q_1) := \int_{\Omega_1} q_1\nabla\cdot\mathbf{v}_1.
$$

To summarize, the variational form is given by

$$
a_{1,\delta}(\mathbf{u}_1,\mathbf{v}_1) - b_1(\mathbf{v}_1,p_1) + \sum_{e\in\Gamma_{12}} \int_e p_2\mathbf{v}_1\cdot\mathbf{n}_{12} = \langle\mathbf{f}_1,\mathbf{v}_1\rangle, \quad \forall\mathbf{v}_1 \in X, \tag{3.2}
$$

$$
b_1(\mathbf{u}_1, q_1) = 0, \quad \forall q_1 \in Q. \tag{3.3}
$$

Now for the single phase flow part on the Darcy domain, we repeat the process with (1.4) and (1.5). It gives

$$
\int_{\Omega_2} q_2\nabla\cdot\mathbf{u}_2 = \int_{\Omega_2} f_2 q_2, \quad \forall q_2 \in Q,
$$

and

$$
\int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_2\cdot\mathbf{v}_2 - \int_{\Omega_2} p_2\nabla\cdot\mathbf{v}_2 - \sum_{e\in\Gamma_{12}} \int_e p_2\mathbf{v}_2\cdot\mathbf{n}_{12} = 0, \quad \forall\mathbf{v}_2 \in X.
$$

Introducing

$$a_2(\mathbf{u}_2, \mathbf{v}_2) := \int_{\Omega_2} \mathbf{K}^{-1}\mathbf{u}_2 \cdot \mathbf{v}_2, \qquad b_2(\mathbf{v}_2, q_2) := \int_{\Omega_2} q_2 \nabla \cdot \mathbf{v}_2,$$

we have

$$a_2(\mathbf{u}_2, \mathbf{v}_2) - b_2(\mathbf{v}_2, p_2) - \sum_{e\in\Gamma_{12}} \int_e p_2 \mathbf{v}_2 \cdot \mathbf{n}_{12} = 0, \quad \forall \mathbf{v}_2 \in X, \tag{3.4}$$

$$b_2(\mathbf{u}_2, q_2) = \langle f_2, q_2 \rangle, \quad \forall q_2 \in Q. \tag{3.5}$$

Further, define

$$a(\mathbf{u}, \mathbf{v}) := a_{1,\delta}(\mathbf{u}_1, \mathbf{v}_1) + a_2(\mathbf{u}_2, \mathbf{v}_2), \qquad b(\mathbf{v}, q) := b_1(\mathbf{v}_1, q_1) + b_2(\mathbf{v}_2, q_2).$$

Since along the interface Γ_{12}, any edge $e = \partial K_1 \cap \Gamma_{12}$ belongs to only one element $K_2 \in \mathcal{T}_h^2$ with $K_1 \in \mathcal{T}_h^1$, we have

$$\sum_{e\in\Gamma_{12}} \int_e p_2(\mathbf{v}_1 \cdot \mathbf{n}_{12} - \mathbf{v}_2 \cdot \mathbf{n}_{12}) = 0.$$

With the above forms, we propose the following variational problem of (1.1)–(1.6): find $(\mathbf{u}, p) \in X \times Q$ such that

$$a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) = \langle \mathbf{f}_1, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in X, \tag{3.6}$$

$$b(\mathbf{u}, q) = \langle f_2, q \rangle, \quad \forall q \in Q. \tag{3.7}$$

Remark 3.1. Note that $a_{1,\delta}(\cdot,\cdot)$ is a symmetric bilinear form for $\delta = -1$. The parameter α is chosen to guarantee coercivity of the form $a(\mathbf{u}, \mathbf{v})$. We assume that α is bounded below by a large enough positive constant in the case of $\delta = -1$.

Theorem 3.1. *If $\mathbf{u} \in V \cap C^2(\bar{\Omega})$, $p \in Q \cap C^2(\bar{\Omega})$, such that $\mathbf{u}|_{\Omega_i} = \mathbf{u}_i$ and $p|_{\Omega_i} = p_i$, then the coupled Stokes and Darcy problem* (1.1)–(1.6) *and the weak problem* (3.6) *and* (3.7) *are equivalent.*

Proof. If $(\mathbf{u}, p)$ solves the coupled Stokes and Darcy problem (1.1)–(1.6), from the process of obtaining Eqs. (3.6) and (3.7), we conclude that $(\mathbf{u}, p)$ is the solution of (3.6) and (3.7). Conversely, if $(\mathbf{u}, p)$ is the solution of (3.6) and (3.7), from the integrating by parts and the regularities of the true solutions, we can prove that $(\mathbf{u}, p)$ solves the coupled Stokes and Darcy problem (1.1)–(1.6). □

4. Finite element approximations

Define X_h and Q_h for the velocity and pressure, respectively, by

$$X_h := \{\mathbf{v}_i \in X : \mathbf{v}_i|_K \in \mathbb{P}_r(K)^2,\ \forall K \in \mathcal{T}_h^i,\ \mathbf{v}_i \cdot \mathbf{n}_i = 0 \text{ on } \Gamma_i, \mathbf{v}_1 \cdot \mathbf{n}_{12} = \mathbf{v}_2 \cdot \mathbf{n}_{12} \text{ on } \Gamma_{12}\},$$

and

$$Q_h := \{q_i \in Q : q_i|_K \in \mathbb{P}_{r-1}(K),\ \forall K \in \mathcal{T}_h^i\},$$

where $r \geq 1$, $\mathbb{P}_m(K)$ is a space of polynomials of degree m on the element K. It holds

$$\nabla \cdot X_h \subset Q_h. \tag{4.1}$$

The approximation solution of (3.6) and (3.7) is: find $\mathbf{u}_h|_{\Omega_i} = \mathbf{u}_{ih} \in X_h$, $p_h|_{\Omega_i} = p_{ih} \in Q_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) = \langle \mathbf{f}_1, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in X_h, \tag{4.2}$$

$$b(\mathbf{u}_h, q) = \langle f_2, q \rangle, \quad \forall q \in Q_h. \tag{4.3}$$

The norm in the linear space X_h on Ω is given by

$$\|\mathbf{v}\|_X^2 = \|\!|\mathbf{v}_1|\!\|_{h,\Omega_1}^2 + \|\mathbf{v}_2\|_{H(\mathrm{div};\Omega_2)}^2, \quad \forall \mathbf{v} \in X_h. \tag{4.4}$$

The norm $\|\!|\mathbf{v}_1|\!\|_{h,\Omega_1}^2$ is defined by

$$\|\!|\mathbf{v}_1|\!\|_{h,\Omega_1}^2 = \|\!|\mathbf{v}_1|\!\|_{1,\Omega_1}^2 + \sum_{e\in\Gamma_h^1\cup\Gamma_1} h_e \|\{\mathbf{D}(\mathbf{v}_1)\mathbf{n}_e \cdot \boldsymbol{\tau}_e\}\|_{0,e}^2, \tag{4.5}$$

with

$$\|\!|\mathbf{v}_1|\!\|_{1,\Omega_1}^2 = \sum_{K\in\mathcal{T}_h^1} \|\mathbf{D}(\mathbf{v}_1)\|_{0,K}^2 + \sum_{e\in\Gamma_h^1\cup\Gamma_1} h_e^{-1} \|[\mathbf{v}_1 \cdot \boldsymbol{\tau}_e]\|_{0,e}^2 + \sum_{e\in\Gamma_{12}} \|\mathbf{v}_1 \cdot \boldsymbol{\tau}_{12}\|_{0,e}^2. \tag{4.6}$$

Here $\|\cdot\|_{0,e}^2 = \int_e |\cdot|^2 \mathrm{d}s$.

Let K be an element with $h_K = \mathrm{diam}(K)$. It is well known that there exists a constant C_{tr} independent of h_K such that (see for example [10])

$$\forall \phi \in H^1(K),\ \forall e \in \partial K, \quad \|\phi\|_{0,e}^2 \le C_{\mathrm{tr}}(h_K^{-1}\|\phi\|_{0,K}^2 + h_K|\phi|_{1,K}^2). \tag{4.7}$$

Observe that the quasi-uniformity of $\mathcal{T}_h$ implies that h_K is proportional to h_e for all the edges $e \subset \partial K$. In particular, one has

$$\forall \mathbf{v} \in X_h,\ \forall e \in \partial K, \quad h_e\|\{\mathbf{D}(\mathbf{v})\mathbf{n}_e \cdot \boldsymbol{\tau}_e\}\|_{0,e}^2 \le C_{\mathrm{tr}}(\|\mathbf{D}(\mathbf{v})\|_{0,K}^2 + h_K^2|\mathbf{D}(\mathbf{v})|_{1,K}^2).$$

The standard inverse inequality can be employed to the last term of the above inequality, which yields

$$\forall \mathbf{v} \in X_h,\ \forall e \in \partial K, \quad h_e\|\{\mathbf{D}(\mathbf{v})\mathbf{n}_e \cdot \boldsymbol{\tau}_e\}\|_{0,e}^2 \le C_{\mathrm{tr}}\|\mathbf{D}(\mathbf{v})\|_{0,K}^2. \tag{4.8}$$

Consequently, there is a constant C_{eq} independent of h such that

$$|||\mathbf{v}_1|||_{h,\Omega_1} \le C_{eq}|||\mathbf{v}_1|||_{1,\Omega_1}, \quad \forall \mathbf{v}_1 \in X_h. \tag{4.9}$$

This shows that the two norms $|||\cdot|||_{1,\Omega_1}$ and $|||\cdot|||_{h,\Omega_1}$ are equivalent in the finite element space X_h.

The norm in the linear space Q_h on Ω is given by

$$\|q\|_Q^2 = \|q_1\|_{0,\Omega_1}^2 + \|q_2\|_{0,\Omega_2}^2, \quad \forall q \in Q_h. \tag{4.10}$$

Let

$$W_h := \{\mathbf{v} \in X_h : \nabla \cdot \mathbf{v}_2 = 0,\ \text{a.e. in } \Omega_2\}.$$

In what follows, we prove the coercivity lemma.

Lemma 4.1. *Suppose that the constant α defined in the bilinear form $a(\cdot,\cdot)$ is large enough such that $\alpha > 2C_{\mathrm{tr}}$ with C_{tr} introduced in (4.8). Then, there exists a positive constant C_X such that*

$$C_X\|\mathbf{v}\|_X^2 \le a(\mathbf{v},\mathbf{v}), \quad \forall \mathbf{v} \in W_h. \tag{4.11}$$

Proof. Let $\mathbf{v} \in W_h$ with $\mathbf{v}|_{\Omega_i} = \mathbf{v}_i$. We have

$$\begin{aligned} a(\mathbf{v},\mathbf{v}) &= a_{1,\delta}(\mathbf{v}_1,\mathbf{v}_1) + a_2(\mathbf{v}_2,\mathbf{v}_2) \\ &= \sum_{K\in\mathcal{T}_h^1}\int_K 2\mu\mathbf{D}(\mathbf{v}_1):\mathbf{D}(\mathbf{v}_1) - \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu(1-\delta)\{\mathbf{D}(\mathbf{v}_1)\,\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\mathbf{v}_1\cdot\boldsymbol{\tau}_e] \\ &\quad + \sum_{e\in\Gamma_{12}}\int_e \frac{\mu}{G}(\mathbf{v}_1\cdot\boldsymbol{\tau}_{12})^2 + \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\alpha h_e^{-1}[\mathbf{v}_1\cdot\boldsymbol{\tau}_e]^2 + \int_{\Omega_2}\mathbf{K}^{-1}(\mathbf{v}_2,\mathbf{v}_2). \end{aligned}$$

The bound on $\mathbf{K}$ in (1.7) gives

$$\int_{\Omega_2}\mathbf{K}^{-1}(\mathbf{v}_2,\mathbf{v}_2) \ge \frac{1}{k_1}\|\mathbf{v}_2\|_{0,\Omega_2}^2.$$

If $\delta = 1$, then the result is straightforward. If $\delta = -1$, from the trace inequality and Young's inequality $ab \le \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$ with $\epsilon = 2C_{\mathrm{tr}}$, we have

$$\begin{aligned} \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu(1-\delta)\{\mathbf{D}(\mathbf{v}_1)\,\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\mathbf{v}_1\cdot\boldsymbol{\tau}_e] &\le 4\mu\sum_{e\in\Gamma_h^1\cup\Gamma_1}\|h^{1/2}\{\mathbf{D}(\mathbf{v}_1)\,\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}\|_{0,e}\|h^{-1/2}[\mathbf{v}_1\cdot\boldsymbol{\tau}_e]\|_{0,e} \\ &\le \frac{4\mu C_{\mathrm{tr}}}{2\epsilon}\sum_{K\in\mathcal{T}_h^1}\|\mathbf{D}(\mathbf{v}_1)\|_{0,K}^2 + \frac{4\mu\epsilon}{2}\sum_{e\in\Gamma_h^1\cup\Gamma_1}h_e^{-1}\|[\mathbf{v}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}^2 \\ &= \mu\sum_{K\in\mathcal{T}_h^1}\|\mathbf{D}(\mathbf{v}_1)\|_{0,K}^2 + 4\mu C_{\mathrm{tr}}\sum_{e\in\Gamma_h^1\cup\Gamma_1}h_e^{-1}\|[\mathbf{v}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}^2. \end{aligned}$$

Thus, for $\delta = -1$ we obtain

$$a(\mathbf{v},\mathbf{v}) \ge \mu\sum_{K\in\mathcal{T}_h^1}\|\mathbf{D}(\mathbf{v}_1)\|_{0,K}^2 + \frac{\mu}{G}\sum_{e\in\Gamma_{12}}\|\mathbf{v}_1\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 + (2\mu\alpha - 4\mu C_{\mathrm{tr}})\sum_{e\in\Gamma_h^1\cup\Gamma_1}h_e^{-1}\|[\mathbf{v}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}^2 + \frac{1}{k_1}\|\mathbf{v}_2\|_{0,\Omega_2}^2.$$

The desired coercivity (4.11) holds true provided that the stabilization parameter α is sufficiently large with $\alpha > 2C_{\mathrm{tr}}$. □

The following are the results on the boundedness of the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$.

Lemma 4.2. *There exist two constants C_a and C_b independent of h such that*

$$|a(\mathbf{u},\mathbf{v})| \le C_a\|\mathbf{u}\|_X\|\mathbf{v}\|_X, \quad \forall \mathbf{u},\mathbf{v}\in X_h, \tag{4.12}$$

and

$$|b(\mathbf{v},q)| \le C_b\|\mathbf{u}\|_X\|q\|_Q, \quad \forall \mathbf{u}\in X_h,\, q\in Q_h. \tag{4.13}$$

Proof. By the definitions of $a_{1,\delta}(\mathbf{u}_1,\mathbf{v}_1)$ and $a_2(\mathbf{u}_2,\mathbf{v}_2)$, and the Schwarz inequality, there exists a constant C_a such that

$$\begin{aligned}
|a_{1,\delta}(\mathbf{u}_1,\mathbf{v}_1)| \le{}& C_a\left\{\left(\sum_{K\in\mathcal{T}_h^1}\|\mathbf{D}(\mathbf{u}_1)\|_{0,K}^2\right)^{\frac12}\left(\sum_{K\in\mathcal{T}_h^1}\|\mathbf{D}(\mathbf{v}_1)\|_{0,K}^2\right)^{\frac12}\right.\\
&+\left(\sum_{e\in\mathcal{T}_h^1\cup\Gamma_1}h_e^{-1}\|[\mathbf{u}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}^2\right)^{\frac12}\left(\sum_{e\in\mathcal{T}_h^1\cup\Gamma_1}h_e^{-1}\|[\mathbf{v}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}^2\right)^{\frac12}\\
&+\left(\sum_{e\in\mathcal{T}_h^1\cup\Gamma_1}h_e^{-1}\|[\mathbf{u}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}^2\right)^{\frac12}\left(\sum_{e\in\mathcal{T}_h^1\cup\Gamma_1}h_e\|\{\mathbf{D}(\mathbf{v}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}\|_{0,e}^2\right)^{\frac12}\\
&+\left(\sum_{e\in\mathcal{T}_h^1\cup\Gamma_1}h_e\|\{\mathbf{D}(\mathbf{u}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}\|_{0,e}^2\right)^{\frac12}\left(\sum_{e\in\mathcal{T}_h^1\cup\Gamma_1}h_e^{-1}\|[\mathbf{v}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}^2\right)^{\frac12}\\
&\left.+\left(\sum_{e\in\Gamma_{12}}\|\mathbf{u}_1\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2\right)^{\frac12}\left(\sum_{e\in\Gamma_{12}}\|\mathbf{v}_1\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2\right)^{\frac12}\right\}\\
\le{}& C_a|||\mathbf{u}_1|||_{h,\Omega_1}|||\mathbf{v}_1|||_{h,\Omega_1},
\end{aligned}$$

and

$$|a_2(\mathbf{u}_2,\mathbf{v}_2)| \le C_a\|\mathbf{u}_2\|_{0,\Omega_2}\|\mathbf{v}_2\|_{0,\Omega_2}.$$

The above two inequalities give that

$$|a(\mathbf{u},\mathbf{v})| \le C_a\|\mathbf{u}\|_X\|\mathbf{v}\|_X$$

which proves the desired boundedness (4.12). Similarly, by the definition of $b(\mathbf{v},q)$ and the Schwarz inequality, there exists a constant C_b such that

$$\begin{aligned}
|b(\mathbf{v},q)| &\le C_b\{\|\nabla\cdot\mathbf{v}_1\|_{0,\Omega_1}\|q_1\|_{0,\Omega_1}+\|\nabla\cdot\mathbf{v}_2\|_{0,\Omega_1}\|q_2\|_{0,\Omega_2}\}\\
&\le C_b\{|||\mathbf{v}_1|||_{h,\Omega_1}\|q_1\|_{0,\Omega_1}+|||\mathbf{v}_2|||_{H(\mathrm{div};\Omega_2)}\|q_2\|_{0,\Omega_2}\}\\
&\le C_b\|\mathbf{u}\|_X\|q\|_Q,
\end{aligned}$$

which is the desired boundness (4.13). □

For any $\theta > 0$ and $\mathbf{v}\in X\cap H^\theta(\Omega_i)^2$, we assume that there exists an interpolation $\Pi_h : X\cap H^\theta(\Omega_i)^2 \to X_h$ satisfying

$$b(\Pi_h\mathbf{v}-\mathbf{v},q)=0, \quad \forall q\in Q_h, \tag{4.14}$$

$$|\Pi_h\mathbf{v}-\mathbf{v}|_{m,K} \le ch_K^{s-m}|\mathbf{v}|_{s,K}, \quad m=\{0,1\},\ 1\le s\le r+1, \tag{4.15}$$

$$\|\nabla\cdot(\Pi_h\mathbf{v}-\mathbf{v})\|_{0,K} \le ch_K^s|\nabla\cdot\mathbf{v}|_{s,K}, \quad 0\le s\le l+1. \tag{4.16}$$

Let $I_h : Q\to Q_h$ be the L^2 orthogonal projection given by

$$\int_\Omega (I_hq-q)w\,\mathrm{d}x = 0, \quad \forall q\in Q,\ \forall w\in Q_h. \tag{4.17}$$

Then, it holds

$$\|q-I_hq\|_{0,K} \le ch_K^s|q|_{s,K}, \quad 0\le s\le l+1. \tag{4.18}$$

Remark 4.1. For some well-known mixed finite element spaces, such as the RT spaces, the BDM spaces and the BDFM spaces, it is known that $\nabla \cdot X_h = Q_h$ and there exists interpolation Π_h satisfying (4.14)–(4.16) (see [22]). Some modified $H(\text{div})$ conforming elements were also proposed in [18] which satisfy the inclusion (4.1) and the conditions (4.14)–(4.16).

For our finite element formulation, a discrete inf – sup condition given in Brezzi's framework is proved as follows.

Lemma 4.3. *There exists a positive constant β independent of h, such that*

$$\sup_{\mathbf{v}_h \in X_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_X \|q_h\|_Q} \geq \beta, \quad \forall q_h \in Q_h. \tag{4.19}$$

Proof. We use Fortin's technique to obtain the results. For $q_h \in Q_h$, there exist a function $\mathbf{v} \in H_0^1(\Omega)^2$ and a constant C_1 such that

$$\nabla \cdot \mathbf{v} = q_h, \qquad \|\mathbf{v}\|_{1,\Omega} \leq C_1 \|q_h\|_{0,\Omega}. \tag{4.20}$$

Inequalities (4.7) and (4.15) imply

$$|||\mathbf{v}_1 - \Pi_h \mathbf{v}_1|||_{1,\Omega_1} \leq c\|\mathbf{v}_1\|_{1,\Omega_1}. \tag{4.21}$$

For $\mathbf{v} \in H_0^1(\Omega)^2$ and $\mathbf{v}|_{\Omega_1} = \mathbf{v}_1$, we have $|||\mathbf{v}_1|||_{1,\Omega_1} \leq c\|\mathbf{v}_1\|_{1,\Omega_1}$. From (4.21) and triangle inequality, it follows that

$$|||\Pi_h \mathbf{v}_1|||_{1,\Omega_1} \leq c\|\mathbf{v}\|_{1,\Omega_1}.$$

Using this inequality and (4.9), we have

$$|||\Pi_h \mathbf{v}_1|||_{h,\Omega_1} \leq C_{eq} |||\Pi_h \mathbf{v}_1|||_{1,\Omega_1} \leq c\|\mathbf{v}_1\|_{1,\Omega_1}.$$

Combing triangle inequality, (4.15) and (4.16), we obtain

$$\begin{aligned} \|\Pi_h \mathbf{v}_2\|_{H(\text{div};\Omega_2)} &\leq \|\Pi_h \mathbf{v}_2 - \mathbf{v}_2\|_{H(\text{div};\Omega_2)} + \|\mathbf{v}_2\|_{H(\text{div};\Omega_2)} \\ &\leq c\|\mathbf{v}_2\|_{1,\Omega_2}. \end{aligned}$$

Note that $\|\mathbf{v}\|_{1,\Omega} := \|\mathbf{v}_1\|_{1,\Omega_1} + \|\mathbf{v}_2\|_{1,\Omega_2}$. From the above two inequalities, we conclude that there exists a constant C_2 such that

$$\|\Pi_h \mathbf{v}\|_X = |||\Pi_h \mathbf{v}_1|||_{h,\Omega_1} + \|\Pi_h \mathbf{v}_2\|_{H(\text{div};\Omega_2)} \leq C_2 \|\mathbf{v}\|_{1,\Omega}. \tag{4.22}$$

We use the operator Π_h to obtain

$$\sup_{\mathbf{v}_h \in X_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_X} \geq \sup_{\mathbf{v} \in H_0^1(\Omega)^2} \frac{b(\Pi_h \mathbf{v}, q_h)}{\|\Pi_h \mathbf{v}\|_X} = \sup_{\mathbf{v} \in H_0^1(\Omega)^2} \frac{b(\mathbf{v}, q_h)}{\|\Pi_h \mathbf{v}\|_X}.$$

Thus, substituting (4.20) and (4.22) into the above inequality gives

$$\sup_{\mathbf{v}_h \in X_h} \frac{b(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_X} \geq C_2^{-1} \sup_{\mathbf{v} \in H_0^1(\Omega)^2} \frac{b(\mathbf{v}, q_h)}{\|\mathbf{v}\|_{1,\Omega}} \geq \beta \|q\|_{0,\Omega}.$$

We then obtain the desired result (4.19) with $\beta = 1/(C_1 C_2)$. □

From Lemmas 4.1–4.3 and Brezzi's theory [22], we easily obtain the following conclusion.

Proposition 4.1. *There exists a unique solution to problems (4.2) and (4.3). Moreover, we have*

$$\|\mathbf{u}_h\|_X + \|p_h\|_Q \leq C(\|\mathbf{f}_1\|_{X'} + \|f_2\|_0). \tag{4.23}$$

5. A priori error estimates

The purpose of this section is to derive a priori error estimates for problems (4.2) and (4.3). To do so, we split the errors $e_{\mathbf{u}} := \mathbf{u} - \mathbf{u}_h$ and $e_p := p - p_h$ into the following forms

$$e_{\mathbf{u}} = (\mathbf{u} - \Pi_h \mathbf{u}) + (\Pi_h \mathbf{u} - \mathbf{u}_h), \qquad e_p = (p - I_h p) + (I_h p - p_h),$$

where $\Pi_h \mathbf{u}$ is the projection of $\mathbf{u}$ defined by (4.14) and $I_h p$ the L^2 projection of p defined by (4.17).

Theorem 5.1. *Let $(\mathbf{u}, p)$ be the solution of the coupled problems (1.1)–(1.6) together with three interface conditions (1.10)–(1.12) such that $\mathbf{u}|_{\Omega_i} \in H^{r+1}(\Omega_i)^2$, $p|_{\Omega_i} \in H^r(\Omega_i)$ for $i = 1, 2$. Then the discrete solution $(\mathbf{u}_h, p_h)$ of problems (4.2) and (4.3) satisfies the error estimate*

$$\|\mathbf{u}-\mathbf{u}_h\|_X+\|p-p_h\|_Q\le ch^r\left(\sum_{i=1}^{2}\|\mathbf{u}_i\|_{r+1,\Omega_i}+\sum_{i=1}^{2}\|p_i\|_{r,\Omega_i}\right). \tag{5.1}$$

Proof. Using the local approximation properties (4.15)–(4.18) and the trace inequality (4.7), we have

$$\begin{aligned}\|\mathbf{u}-\Pi_h\mathbf{u}\|_X &= (|\!|\!|\mathbf{u}_1-\Pi_h\mathbf{u}_1|\!|\!|^2_{h,\Omega_1}+\|\mathbf{u}_2-\Pi_h\mathbf{u}_2\|^2_{H(\mathrm{div};\Omega_2)})^{1/2}\\ &\le c(h^{2r}\|\mathbf{u}_1\|^2_{r+1,\Omega_1}+h^{2r}\|\mathbf{u}_2\|^2_{r+1,\Omega_2})^{1/2}\\ &\le ch^r(\|\mathbf{u}_1\|_{r+1,\Omega_1}+\|\mathbf{u}_2\|_{r+1,\Omega_2}),\end{aligned} \tag{5.2}$$

and

$$\begin{aligned}\|p-I_hp\|_Q &= (\|p_1-I_hp_1\|^2_{0,\Omega_1}+\|p_2-I_hp_2\|^2_{0,\Omega_2})^{1/2}\\ &\le ch^r(\|p_1\|_{r,\Omega_1}+\|p_2\|_{r,\Omega_2}).\end{aligned} \tag{5.3}$$

In view of the projection approximation properties (5.2) and (5.3), we only need to estimate the errors $\mathbf{u}_h-\Pi_h\mathbf{u}:=\boldsymbol{\chi}$ and $p_h-I_hp:=\xi$ with $\boldsymbol{\chi}|_{\Omega_i}=\boldsymbol{\chi}_i$ and $\xi|_{\Omega_i}=\xi_i$.

From (3.6), (3.7), (4.2) and (4.3), the error equation satisfy

$$a(\boldsymbol{\chi},\mathbf{v})-b(\mathbf{v},\xi)=a(\mathbf{u}-\Pi_h\mathbf{u},\mathbf{v})-b(\mathbf{v},p-I_hp),\quad \forall\mathbf{v}\in X_h, \tag{5.4}$$

$$b(\boldsymbol{\chi},q)=b(\mathbf{u}-\Pi_h\mathbf{u},q),\quad \forall q\in Q_h. \tag{5.5}$$

Note that (4.14) implies that $b(\mathbf{u}-\Pi_h\mathbf{u},q)=0$ for all $q\in Q_h$. Choose $\mathbf{v}=\boldsymbol{\chi}$ and $q=\xi$, then $b(\boldsymbol{\chi},\xi)=0$ and

$$a(\boldsymbol{\chi},\boldsymbol{\chi})=a(\mathbf{u}-\Pi_h\mathbf{u},\boldsymbol{\chi})+b(\boldsymbol{\chi},p-I_hp). \tag{5.6}$$

Define $\tilde{\mathbf{u}}=\mathbf{u}-\Pi_h\mathbf{u}$ with $\tilde{\mathbf{u}}|_{\Omega_i}=\tilde{\mathbf{u}}_i$. The first term $a_{1,\delta}(\tilde{\mathbf{u}}_1,\boldsymbol{\chi}_1)$ of $a(\cdot,\cdot)$ on the right-hand side of (5.6) can be estimated as follows

$$\begin{aligned}a_{1,\delta}(\tilde{\mathbf{u}}_1,\boldsymbol{\chi}_1) &= \sum_{K\in\mathcal{T}_h^1}\int_K 2\mu\mathbf{D}(\tilde{\mathbf{u}}_1):\mathbf{D}(\boldsymbol{\chi}_1)-\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\{\mathbf{D}(\tilde{\mathbf{u}}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]\\ &\quad+\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\alpha h_e^{-1}[\tilde{\mathbf{u}}_1\cdot\boldsymbol{\tau}_e][\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]+\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\delta\{\mathbf{D}(\boldsymbol{\chi}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\tilde{\mathbf{u}}_1\cdot\boldsymbol{\tau}_e]\\ &\quad+\sum_{e\in\Gamma_{12}}\int_e\frac{\mu}{G}(\tilde{\mathbf{u}}_1\cdot\boldsymbol{\tau}_{12})(\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_{12}):=T_1+\cdots+T_5.\end{aligned}$$

Using Cauchy–Schwarz inequality and (4.15), we have

$$T_1\le 2\mu\sum_T\|\mathbf{D}(\tilde{\mathbf{u}}_1)\|_{0,T}\|\mathbf{D}(\boldsymbol{\chi}_1)\|_{0,T}\le\frac{1}{8}|\boldsymbol{\chi}_1|_{1,\Omega_1}+Ch^{2r}\|\mathbf{u}_1\|^2_{r+1,\Omega_1}. \tag{5.7}$$

Since we can only obtain the zeroth-order and first-order seminorm estimates of $\tilde{\mathbf{u}}_1$, we use the same skills as in [10] to bound the second term T_2. Let $L_h(\mathbf{u}_1)$ denote the standard Lagrange interpolant of degree r in Ω_1. Note that $L_h(\mathbf{u}_1)$ satisfies the optimal approximation

$$|\mathbf{u}_1-L_h(\mathbf{u}_1)|_{m,K}\le Ch_K^{s-m}|\mathbf{u}_1|_{r+1,K},\quad 2\le s\le r+1,\ m=0,1,2. \tag{5.8}$$

For e a segment of $\Gamma_h^1\cup\Gamma_1$, we have

$$\int_e\{\mathbf{D}(\tilde{\mathbf{u}}_1)\,\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]=\int_e\{\mathbf{D}(\mathbf{u}_1-L_h(\mathbf{u}_1))\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]+\int_e\{\mathbf{D}(L_h(\mathbf{u}_1)-\Pi_h\mathbf{u}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e].$$

From Cauchy–Schwarz inequality, the trace inequality (4.7) and (5.8), we have

$$\begin{aligned}&\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\{\mathbf{D}(\mathbf{u}_1-L_h(\mathbf{u}_1))\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\boldsymbol{\chi}\cdot\boldsymbol{\tau}_e]\le 2\mu\sum_{e\in\Gamma_h^1\cup\Gamma_1}h_e^{-1/2}\|[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}h_e^{1/2}\|\{\mathbf{D}(\mathbf{u}_1-L_h(\mathbf{u}_1))\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}\|_{0,e}\\ &\quad\le\frac{1}{16}\sum_{e\in\Gamma_h^1\cup\Gamma_1}h_e^{-1}\|[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]\|^2_{0,e}+C\sum_{K\in\mathcal{T}_h^1}\int_K h_K(h_K^{-1}\|\mathbf{D}(\mathbf{u}_1-L_h(\mathbf{u}_1))\|^2_{0,K}+h_K|\mathbf{D}(\mathbf{u}_1-L_h(\mathbf{u}_1))|^2_{1,K})\\ &\quad\le\frac{1}{16}|\!|\!|\boldsymbol{\chi}_1|\!|\!|^2_{1,\Omega_1}+Ch^{2r}\|\mathbf{u}_1\|^2_{r+1,\Omega_1}.\end{aligned}$$

The last two inequalities hold because h_K is proportional to h_e for all the edges $e \subset \partial K$. Similarly, in addition to using the inverse inequality and triangular inequality, we have

$$\begin{aligned}
&\sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\{\mathbf{D}(L_h(\mathbf{u}_1)-\Pi_h\mathbf{u}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e] \le 2\mu\sum_{e\in\Gamma_h^1\cup\Gamma_1} h_e^{-1/2}\|[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}h_e^{1/2}\|\{\mathbf{D}(L_h(\mathbf{u}_1)-\Pi_h\mathbf{u}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}\|_{0,e}\\
&\le \frac{1}{16}\sum_{e\in\Gamma_h^1\cup\Gamma_1} h_e^{-1}\|[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}^2 + C\sum_{K\in\mathcal{T}_h^1}\int_K h_K(h_K^{-1}\|\mathbf{D}(L_h(\mathbf{u}_1)-\Pi_h\mathbf{u}_1)\|_{0,K}^2 + h_K|\mathbf{D}(L_h(\mathbf{u}_1)-\Pi_h\mathbf{u}_1)|_{1,K}^2)\\
&\le \frac{1}{16}\sum_{e\in\Gamma_h^1\cup\Gamma_1} h_e^{-1}\|[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}^2 + C\sum_{K\in\mathcal{T}_h^1}\int_K \|\mathbf{D}(L_h(\mathbf{u}_1)-\Pi_h\mathbf{u}_1)\|_{0,K}^2\\
&\le \frac{1}{16}|\!|\!|\boldsymbol{\chi}_1|\!|\!|_{1,\Omega_1}^2 + Ch^{2r}\|\mathbf{u}_1\|_{r+1,\Omega_1}^2.
\end{aligned}$$

Combining the above two inequalities gives

$$\begin{aligned}
T_2 &= \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\{\mathbf{D}(\tilde{\mathbf{u}}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]\\
&\le \frac{1}{8}|\!|\!|\boldsymbol{\chi}_1|\!|\!|_{1,\Omega_1}^2 + Ch^{2r}\|\mathbf{u}_1\|_{r+1,\Omega_1}^2. \qquad (5.9)
\end{aligned}$$

Using Cauchy–Schwarz inequality, the trace inequality (4.7) and the approximation result (4.15), we have

$$\begin{aligned}
T_3 &= \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\alpha h_e^{-1}[\tilde{\mathbf{u}}_1\cdot\boldsymbol{\tau}_e][\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]\\
&\le 2\mu\alpha\sum_{e\in\Gamma_h^1\cup\Gamma_1} h_e^{-1/2}\|[\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}h_e^{-1/2}\|[\tilde{\mathbf{u}}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}\\
&\le \frac{1}{8}|\!|\!|\boldsymbol{\chi}_1|\!|\!|_{1,\Omega_1}^2 + Ch^{2r}\|\mathbf{u}_1\|_{r+1,\Omega_1}^2. \qquad (5.10)
\end{aligned}$$

Similarly, the terms T_4 and T_5 are bounded as

$$\begin{aligned}
T_4 &= \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\delta\{\mathbf{D}(\boldsymbol{\chi}_1)\,\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}[\tilde{\mathbf{u}}_1\cdot\boldsymbol{\tau}_e]\\
&\le \sum_{e\in\Gamma_h^1\cup\Gamma_1}\int_e 2\mu\delta h_e^{1/2}\|\{\mathbf{D}(\boldsymbol{\chi}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}\|_{0,e}h_e^{-1/2}\|[\tilde{\mathbf{u}}_1\cdot\boldsymbol{\tau}_e]\|_{0,e}\\
&\le \frac{1}{8}\sum_{e\in\Gamma_h^1\cup\Gamma_1} h_e\|\{\mathbf{D}(\boldsymbol{\chi}_1)\mathbf{n}_e\cdot\boldsymbol{\tau}_e\}\|_{0,e}^2 + Ch^{2r}\|\mathbf{u}_1\|_{r+1,\Omega_1}^2, \qquad (5.11)
\end{aligned}$$

and

$$\begin{aligned}
T_5 &= \sum_{e\in\Gamma_{12}}\int_e \frac{\mu}{G}(\tilde{\mathbf{u}}_1\cdot\boldsymbol{\tau}_{12})(\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_{12})\\
&\le \frac{\mu}{G}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_{12}\|_{0,e}\|\tilde{\mathbf{u}}_1\cdot\boldsymbol{\tau}_{12}\|_{0,e}\\
&\le \frac{1}{16}\sum_{e\in\Gamma_{12}}\|\boldsymbol{\chi}_1\cdot\boldsymbol{\tau}_{12}\|_{0,e}^2 + Ch^{2r+1}\|\mathbf{u}_1\|_{r+1,\Omega_1}^2. \qquad (5.12)
\end{aligned}$$

Let us now estimate $a_2(\tilde{\mathbf{u}}_2, \boldsymbol{\chi}_2)$ of $a(\cdot,\cdot)$ on the right-hand side of (5.6). Using the bound on $\mathbf{K}$ in (1.7) and the approximation result (4.17), we have

$$\begin{aligned}
a_2(\tilde{\mathbf{u}}_2, \boldsymbol{\chi}_2) &= \int_{\Omega_2}\mathbf{K}^{-1}\tilde{\mathbf{u}}_2\cdot\boldsymbol{\chi}_2 \le \|\boldsymbol{\chi}_2\|_{0,\Omega_2}\|\mathbf{K}^{-1}\tilde{\mathbf{u}}_2\|_{0,\Omega_2}\\
&\le \frac{1}{16}\|\boldsymbol{\chi}_2\|_{0,\Omega_2}^2 + Ch^{2r+2}\|\mathbf{u}_2\|_{r+1,\Omega_2}^2. \qquad (5.13)
\end{aligned}$$

It remains to estimate $b(\boldsymbol{\chi}, p - I_h p)$ on the right-hand side of (5.6). Using the inclusion $\nabla \cdot X_h \subset Q_h$ and property (4.17) of the operator I_h, we have

$$b(\boldsymbol{\chi}, p - I_h p) = \int_{\Omega_1} (p_1 - I_h p_1)\nabla \cdot \boldsymbol{\chi}_1 + \int_{\Omega_2} (p_2 - I_h p_2)\nabla \cdot \boldsymbol{\chi}_2$$
$$= 0. \tag{5.14}$$

Combining all bounds above yields

$$a(\boldsymbol{\chi}, \boldsymbol{\chi}) \le \frac{9}{16} |||\boldsymbol{\chi}_1|||^2_{1,\Omega_1} + \frac{1}{16} \|\boldsymbol{\chi}_2\|^2_{H(\mathrm{div},\Omega_2)} + C\left(h^{2r} \sum_{i=1}^{2} \|\mathbf{u}_i\|^2_{r+1,\Omega_i} + h^{2r} \sum_{i=1}^{2} \|p_i\|^2_{r,\Omega_i}\right). \tag{5.15}$$

Combing triangle inequality, Lemma 4.1 and (5.15), we have

$$\|\mathbf{u} - \mathbf{u}_h\|_X \le ch^r \left(\sum_{i=1}^{2} \|\mathbf{u}_i\|_{r+1,\Omega_i} + \sum_{i=1}^{2} \|p_i\|_{r,\Omega_i}\right). \tag{5.16}$$

The error equation (5.4) can be written as

$$b(\mathbf{v}, I_h p - p_h) = a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p - I_h p), \quad \forall \mathbf{v} \in X_h. \tag{5.17}$$

For the first term on the right-hand side of (5.17), we have

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) = &\sum_{K \in \mathcal{T}_h^1} \int_K 2\mu \mathbf{D}(\mathbf{u}_1 - \mathbf{u}_{1h}) : \mathbf{D}(\mathbf{v}_1) - \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e 2\mu \{\mathbf{D}(\mathbf{u}_1 - \mathbf{u}_{1h})\, \mathbf{n}_e \cdot \boldsymbol{\tau}_e\}[\mathbf{v}_1 \cdot \boldsymbol{\tau}_e] \\ &+ \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e 2\mu\alpha h_e^{-1} [(\mathbf{u} - \mathbf{u}_h) \cdot \boldsymbol{\tau}_e][\mathbf{v}_1 \cdot \boldsymbol{\tau}_e] + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e 2\mu\delta \{\mathbf{D}(\mathbf{v}_1)\, \mathbf{n}_e \cdot \boldsymbol{\tau}_e\}[(\mathbf{u}_1 - \mathbf{u}_{1h}) \cdot \boldsymbol{\tau}_e] \\ &+ \sum_{e \in \Gamma_{12}} \int_e \frac{\mu}{G} ((\mathbf{u}_1 - \mathbf{u}_{1h}) \cdot \boldsymbol{\tau}_{12})(\mathbf{v}_1 \cdot \boldsymbol{\tau}_{12}) + \int_{\Omega_2} \mathbf{K}^{-1} (\mathbf{u}_2 - \mathbf{u}_{2h}) \cdot \mathbf{v}_2 \\ := &\; Q_1 + \cdots + Q_6. \end{aligned}$$

We now estimate each Q_i term. From Cauchy–Schwarz inequality, the terms Q_1, Q_3, Q_5 and Q_6 can be easily bounded as

$$Q_1 + Q_3 + Q_5 + Q_6 \le c\|\mathbf{v}\|_X \|\mathbf{u} - \mathbf{u}_h\|_X. \tag{5.18}$$

For the term Q_2, we have

$$\begin{aligned} |Q_2| &= \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e 2\mu \{\mathbf{D}(\mathbf{u}_1 - \mathbf{u}_{1h})\, \mathbf{n}_e \cdot \boldsymbol{\tau}_e\}[\mathbf{v}_1 \cdot \boldsymbol{\tau}_e] \\ &\le c \sum_{e \in \Gamma_h^1 \cup \Gamma_1} h_e^{-1/2} \|[\mathbf{v}_1 \cdot \boldsymbol{\tau}_e]\|_{0,e} h_e^{1/2} \|\{\mathbf{D}(\mathbf{u}_1 - \mathbf{u}_{1h})\mathbf{n}_e \cdot \boldsymbol{\tau}_e\}\|_{0,e} \\ &\le c\|\mathbf{v}\|_X \left(\sum_{e \in \Gamma_h^1 \cup \Gamma_1} (h_e \|\{\mathbf{D}(\mathbf{u}_{1h} - \Pi_h \mathbf{u}_{1h})\mathbf{n}_e \cdot \boldsymbol{\tau}_e\}\|^2_{0,e} + h_e \|\{\mathbf{D}(\mathbf{u}_1 - \Pi_h \mathbf{u}_{1h})\, \mathbf{n}_e \cdot \boldsymbol{\tau}_e\}\|^2_{0,e})\right)^{1/2} \\ &\le c\|\mathbf{v}\|_X (\|\Pi_h \mathbf{u} - \mathbf{u}_h\|^2_X + ch^{2r} \|\mathbf{u}_1\|^2_{r+1,\Omega_1})^{1/2}. \end{aligned}$$

Similarly, from the trace inequality (4.7) the term Q_4 is bounded as

$$\begin{aligned} Q_4 &= \sum_{e \in \Gamma_h^1 \cup \Gamma_1} \int_e 2\mu\delta \{\mathbf{D}(\mathbf{v}_1)\mathbf{n}_e \cdot \boldsymbol{\tau}_e\}[(\mathbf{u}_1 - \mathbf{u}_{1h}) \cdot \boldsymbol{\tau}_e] \\ &\le c \sum_{e \in \Gamma_h^1 \cup \Gamma_1} h_e^{1/2} \|\{\mathbf{D}(\mathbf{v}_1)\mathbf{n}_e \cdot \boldsymbol{\tau}_e\}\|_{0,e} h_e^{-1/2} \|[(\mathbf{u}_1 - \mathbf{u}_{1h}) \cdot \boldsymbol{\tau}_e]\|_{0,e} \\ &\le c\|\mathbf{v}\|_X \|\mathbf{u} - \mathbf{u}_h\|_X. \end{aligned}$$

The remaining is to estimate $b(\mathbf{v}, p - I_h p)$ on the right-hand side of (5.6). Using the inclusion $\nabla \cdot X_h \subset Q_h$ and property (4.17) of the operator I_h, we have

$$b(\mathbf{v}, p - I_h p) = \int_{\Omega} (p - I_h p)\nabla \cdot \mathbf{v} = 0.$$

From the discrete inf – sup condition (4.19), we have

$$\|I_h p - p_h\|_Q \le \frac{1}{\beta} \sup_{\mathbf{v}\in X_h} \frac{b(\mathbf{v}, I_h p - p_h)}{\|\mathbf{v}\|_X}. \tag{5.19}$$

Combining all the bounds above, (5.17), (5.19) and triangle inequality yields

$$\|p - p_h\|_Q \le c \left(\|\mathbf{u} - \mathbf{u}_h\|_X + \|\Pi_h \mathbf{u} - \mathbf{u}_h\|_X + h^r \sum_{i=1}^{2} \|\mathbf{u}_i\|_{r+1,\Omega_i} + h^r \sum_{i=1}^{2} \|p_i\|_{r,\Omega_i} \right).$$

Using (5.2), (5.3) and (5.16) concludes the proof. □

6. Conclusions

In this paper, we have used $H(\text{div})$ conforming finite elements for the discretization of the coupled Stokes and Darcy problem, and established optimal a prior estimates for the velocity and pressure approximation. This method can naturally deal with the interface conditions.

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